From Skyrms, Choice and Chance

VII.4. THE LOGICAL INTERPRETATION OF PROBABILITY.

The logical interpretation of probability is intended to account for the inductive use of "probability." Accordingly it takes the probability calculus to be formulated in terms of statements rather than properties. On the logical interpretation, conditional probabilities express *logical relations* between the statements involved (although these logical relations are different from the ones encountered in the study of deductive logic). Of course, in one sense it is obvious that, in the inductive use of "probability," $\Pr(q \text{ given } p)$ must express a logical relation between p and q. That is, it grades the inductive strength of that argument which has p as its premise and q as its conclusion.

However, something additional is meant by the claim that conditional probabilities express logical relations between the statements involved. What is meant is that probability ascription statements, that is, $\Pr(q \text{ given } p) = a$, are *analytic*; their truth or falsity does not depend on the facts. Under the logical interpretation, the value of $\Pr(q \text{ given } p)$ depends solely on the meaning of "probability" and the meanings of the statements "p" and "q" and is independent of the facts. Consequently no empirical investigation would be relevant to determining probability values, just as no empirical investigation would be relevant to determining the truth or falsity of the statement $p \vee \sim p$. To put the matter another way, under the logical interpretation, probability ascription statements, $\Pr(q \text{ given } p) = a$, make no factual claim. In contrast, under the relative frequency interpretation, such statements do make a factual claim about a relative frequency or limit of a relative frequency. Consequently under the relative frequency interpretation, empirical investigation would be relevant to determining probability values.

Those who maintain that a logical interpretation, rather than a frequency interpretation, is necessary to account for the inductive use of probability reason as follows: The inductive probability, $\Pr(q \text{ given } p)$, grades the evidential support that p gives to q. But if probability statements such as $\Pr(q \text{ given } p) = a$ are interpreted as making factual claims, then they must be evaluated inductively on the basis of the available evidence, e. To do this, we must use the inductive probability, $\Pr[\Pr(q \text{ given } p) = a \text{ given } e]$. But the ascription of a value to this inductive probability will also make a factual claim, and in order to evaluate this factual claim we must know another inductive probability, and so on *ad infinitum*. In order to know the value of an inductive probability, we would have to already know the value of an infinite number of other inductive probabilities. If this is the case, inductive logic could never get off the ground. Thus the proponent of the logical interpretation reasons that the choice of any other interpretation to account for the inductive use of probability leads to an infinite regress.

It should not be thought that, because probability ascriptions are analytic under the logical interpretation, the logical interpretation cannot take account of empirical data. The statement $\Pr(q \text{ given } p) = a$ will be analytic, but the statements p and q may make factual claims. (Indeed they may be statements about relative frequencies; thus the logical interpretation allows us to use inductive probability in order to evaluate the evidential support that a statement about relative frequency gives another statement, and vice versa.) To put the matter another way, under the logical interpretation, the epistemic probability of a statement, p, depends on just what empirical data is in our stock of knowledge. Thus while $\Pr_a(q \text{ given } p) = a$ would be analytic, $\Pr_e(q) = a$ would not.

As an illustration of a logical interpretation of probability, we shall use a simple version of an interpretation suggested by Carnap. Consider a simple language that has only two names, a and b, and two (logically independent) properties, F and G. The simple statements of this language are Fa; Fb; Ga; Gb. Complex statements are constructed out of simple statements by means of the logical connectives, as are complex properties. A *state description* is a conjunction containing as its conjuncts each atomic statement or its negation, but not both. The state descriptions for our language are:

1. $Fa\&Ga\&Fb\&Gb$.	9. $\sim Fa\&Ga\&Fb\&Gb$.
2. $Fa\&Ga\&Fb\&\sim Gb$.	10. $\sim Fa\&Ga\&Fb\&\sim Gb$.
3. $Fa\&Ga\&\sim Fb\&Gb$.	11. $\sim Fa\&Ga\&\sim Fb\&Gb$.
4. $Fa\&Ga\&\sim Fb\&Gb$.	12. $\sim Fa\&Ga\&\sim Fb\&\sim Gb$.
5. $Fa\&\sim Ga\&Fb\&Gb$.	13. $\sim Fa\&\sim Ga\&Fb\&Gb$.
6. $Fa\&\sim Ga\&Fb\&\sim Gb$.	14. $\sim Fa\&\sim Ga\&Fb\&\sim Gb$.
7. $Fa\&\sim Ga\&\sim Fb\&Gb$.	15. $\sim Fa\&\sim Ga\&\sim Fb\&Gb$.
8. $Fa\&\sim Ga\&\sim Fb\&\sim Gb$.	16. $\sim Fa\&\sim Ga\&\sim Fb\&\sim Gb$.

Each state description describes a possible state of affairs with respect to what properties are had by a and b. We will say that a statement holds in a given state description if and only if the argument with that state description as its only premise and that statement as its conclusion is deductively valid. For instance, Fa holds in state descriptions 1 through 8. A tautology, such as $Fa \vee \sim Fa$, holds in all state descriptions. A contradiction, such as $Fa\&\sim Fa$, holds in no state descriptions. Now we could give a logical interpretation of the probability calculus as follows:

$$\mathrm{Pr}_a(p) = \frac{\text{The number of state descriptions in which } p \text{ holds}}{\text{The total number of state descriptions}}$$

$$\mathrm{Pr}_a(q \text{ given } p) = \frac{\mathrm{Pr}_a(p\&q)}{\mathrm{Pr}_a(p)}$$

At first glance, this interpretation seems quite plausible. It gives each of the state descriptions equal *a priori* probability. However, Carnap has shown that this interpretation has disastrous consequences for inductive logic. If we accept this interpretation then it will always be the case that $\mathrm{Pr}_a(q) = \mathrm{Pr}_a(q \text{ given } p)$.[4] Consequently the epistemic probability of q will always be equal to its *a priori* probability and will not be influenced by the stock of knowledge, e. Any interpretation of the probability calculus which leads to the result that the stock of knowledge is irrelevant to the epistemic probability of a statement cannot account for the inductive use of "probability." We must look elsewhere for a logical interpretation that can account for this inductive use.

Consider those complex properties that are conjunctions, containing as their conjuncts each simple property or its negation but not both. These are called Q properties. The Q properties in the language under consideration are:

Q_1: $F\&G$
Q_2: $\sim F\&G$
Q_3: $F\&\sim G$
Q_4: $\sim F\&\cancel{G}$ ~G

Note that in a presence table each Q property is present in exactly one case, and in each case exactly one Q property is present:

	F	G	Q_1	Q_2	Q_3	Q_4
Case 1:	P	P	P	A	A	A
Case 2:	A	P	A	P	A	A
Case 3:	P	A	A	A	P	A
Case 4:	A	A	A	A	A	P

The state descriptions can be easily reformulated in terms of Q properties. For instance, state description 1 becomes $Q_1a\&Q_1b$, state description 2 becomes $Q_1a\&Q_3b$, state description 3 becomes $Q_1a\&Q_2b$, etc.

[4] For simple (atomic) statements p and q.

We will call a statement that specifies how many individuals have each Q property a *structure description.* For instance, the following is a structure description: One individual has Q_1 and one individual has Q_4 and no individuals have Q_2 or Q_3. It is possible for more than one state description to correspond to a given structure description. For instance, state descriptions 4 and 13 correspond to the structure description given (since 4 can be reformulated as $Q_1a\&Q_4b$ and 13 can be reformulated as $Q_4a\&Q_1b$). State descriptions that correspond to the same structure description are said to be *isomorphic* to each other. On the other hand, some structure descriptions may have only one corresponding state description. For instance, only state description 1 corresponds to the structure description: Two individuals have Q_1 and no individuals have Q_2, Q_3, or Q_4. The 10 structure descriptions and the state descriptions that correspond to them are tabulated in Table 1.

Table 1

Structure description	Corresponding state description
Two Q_1	1
Two Q_2	11
Two Q_3	6
Two Q_4	16
One Q_1 and one Q_2	3, 9
One Q_1 and one Q_3	2, 5
One Q_1 and one Q_4	4, 13
One Q_2 and one Q_3	7, 10
One Q_2 and one Q_4	12, 15
One Q_3 and one Q_4	8, 14

Now instead of assigning *state descriptions* equal *a priori* probabilities, we shall assign *structure descriptions* equal *a priori* probabilities. In our illustration, we assign each structure description a value of $\frac{1}{10}$. When there is only one state description corresponding to a structure description, we assign it the probability value assigned to that structure description. Thus state descriptions 1, 11, 6, and 16 will each have an *a priori* probability of $\frac{1}{10}$. When more than one state description corresponds to a given structure description, we will divide the probability assigned to that structure description equally among the corresponding state descriptions. Therefore all the other state descriptions in our illustration will have an *a priori* probability $\frac{1}{20}$. Finally we will let the *a priori* probability of a statement be the sum of the *a priori* probabilities of all the state descriptions in which that statement holds. Along these lines we advance the following interpretation of the probability calculus in a given language, L:

> Let S be a state description in L. Let a be the number of structure descriptions in L and b be the number of state descriptions isomorphic to S. Then
>
> $$\Pr_a(S) = \frac{1}{a \times b}$$
>
> For any statement p in L, $\Pr_a(p)$ = the sum of the probabilities if the state descriptions in which p holds. (If p is a self-contradiction, and thus holds in no state description, $\Pr_a(p) = 0$.)

Conditional probabilities are defined in the usual way. This interpretation of the probability calculus avoids the difficulties of the earlier one. It allows our body of knowledge to influence epistemic probabilities in a reasonable manner. However, it is far from perfect, for reasons we will not go into here. The search for a plausible logical interpretation of probability has covered some interesting terrain. If you want to find out about it, try the following:

Suggested readings

Rudolf Carnap, *Logical Foundations of Probability*, 2nd ed. (Chicago: University of Chicago Press, 1962), chap. III and appendix.

Rudolf Carnap, *The Continuum of Inductive Methods* (Chicago: University of Chicago Press, 1952), pp. 1–55.

Jaako Hintikka, "A Two Dimensional Continuum of Inductive Methods," in *Aspects of Inductive Logic*, ed. Hintikka and Suppes (New York: Humanities Press, 1966).

VII.5 THE BAYESIAN INTERPRETATION. Bayesians view the probability calculus as a set of *rules of rationality* or *consistency conditions* for degrees of belief or for the betting behavior correlated with degrees of belief. The reason for calling these *consistency* conditions lies in the Dutch-Book theorems of Chapter VI. Let us call a set of betting quotients on a field of statements *coherent* if it admits of no

Dutch Book. Then Chapter VI showed that any coherent set of betting quotients for an appropriate field of propositions is an interpretation of the probability calculus.

The classical Bayesian position is that the rules of the probability calculus are the *only* such consistency conditions for degrees of belief. This position stands in sharp contrast to that which follows from the logical interpretation. According to the logical interpretation, there is one, analytically true, set of inductive probability statements. A rational set of epistemic probabilities would then be one generated by the true inductive probabilities operating on some stock of knowledge.

There is, however, a middle ground toward which many investigators have been moving. On the logical side it may be conceded that the meaning of inductive probability may not single out just one set of inductive probabilities (confirmation function), but rather a range of them. On the other hand, a Bayesian may admit that the rules of the probability calculus alone are not enough to fully account for the inductive aspects of degrees of belief. Remember Carnap's observation that the probability distribution which assigns each state description equal probability does not allow learning from experience in the following sense. If p is an atomic statement, and we move from the probability distribution in question to a new one by conditionalizing on an atomic sentence different from p, then $\Pr(p)$ will remain the same. Likewise for a long series of such conditionalizations. (A similar result holds for Jeffrey's rule.) This would mean the *only* way of changing our degree of belief in p would be to observe it. Such a set of epistemic probabilities is an inductive *trap* and might be considered irrational on just these grounds. Narrowing down the permissible epistemic probability distributions narrows down the possible inductive probabilities which could have generated them.

Thus, proponents of both the logical and Bayesian views may find themselves pursuing the same enterprise from different standpoints. Both regard the Dutch-Book theorem as showing that rational degrees of belief are probabilities. Both are interested in finding further constraints on rational degrees of belief and rational change of belief.

Exercise:

Speaking of applying the rule of conditionalization to the probability distribution which gives equal probability to each state description, I said that a similar result holds for Jeffrey's rule. What is that result?