Letters to the Editor

DISCUSSION OF PIEGORSCH AND GLADEN (1986)

Piegorsch and Gladen (1986) proposed a method for constructing interval estimates for the mean of a gamma distribution for cases in which there is some prior interval information as to the location of the mean parameter. Although in total agreement with the authors' objective—the incorporation of prior information—I feel that the numerical difficulties that they encountered are largely of their own making.

Before discussing their numerical difficulties, I want to correct the authors' algorithm for choosing the parameters of an inverse gamma prior distribution for the mean parameter θ. The authors wished to find values of a and b that satisfy

$$\int_L^U \frac{b^{-a}\theta^{-(a+1)}e^{-1/(\theta b)}\,d\theta}{\Gamma(a)} = p_0 \qquad (1)$$

with equal probabilities $(1 - p_0)/2$ below and above L and U, respectively. Making the transformation $x = \theta^{-1}$, (1) may be written as

$$\int_{1/U}^{1/L} \frac{b^{-a}x^{a-1}e^{x/b}\,dx}{\Gamma(a)} = p_0. \qquad (2)$$

Comparison of (2) with (6.9) in Martz and Waller (1982) shows that the parameters are related as follows:

$$a = \alpha_0, \quad b = 1/\beta_0, \quad U = 1/LL, \quad L = 1/UL.$$

Substituting these values in step 5 of Martz and Waller (1982, p. 239), it is clear that the authors' Step 5 is incorrect and should read:

Step 5. Calculate $b = 1/(b_0 U \times 10^{-6})$.

The correct values for a and b corresponding to the authors' choices for L, U and p_0 are shown in Table 1. The values in the table were obtained by solving (1) exactly, subject to the equal-tail-probabilities condition. Values close to these may be obtained, however, by using the tables and nomograms in appendix C of Martz and Waller (1982).

Turning now to numerical questions, the basic problem is to evaluate the following integral:

$$p(\theta \mid t, v) \propto e^{-1/(\theta b)}\theta^{-(a+1)} \times \int_1^{\Psi} v^r r^{nr} e^{-tr/\theta}\,dr/(\Gamma^n(r)\theta^{nr}), \quad (3)$$

which the authors obtained by assuming an inverse gamma prior with parameters a and b for θ and independently a uniform prior for r between 1 and Ψ. Of this integral the authors said: "In practice we found that, even for fairly low values of n, the integrand in (2.2) proved highly unstable" (p. 270); "By employing quadruple precision floating point arithmetic, we were able to achieve some computational stability..." (pp. 270–271); "Quadruple precision floating point calculations are fairly expensive ... because of this, and because of the greater question of numerical stability we cannot recommend use of the inverse gamma prior formulation" (p. 271); and "the computational vagaries involved with the use of (2.2) make its usefulness highly questionable" (p. 272).

Although I can agree with all of these statements as they apply to the authors' example and prior specification, I would suggest that they are the result of an unhappy choice of the prior, not for θ, but for r. It is not hard to show that the posterior distribution of r has the form

$$p(r \mid t, v) \propto \frac{v^r r^{nr}\Gamma(a + nr)b^{a+nr}}{\Gamma^n(r)(1 + trb)^{a+nr}}, \qquad 1 < r < \Psi, \quad (4)$$

and although the mode of this distribution is not analytically available, for given v, t, a, and b it may be found numerically. For all choices of a and b in the authors' table 2, the mode of (4) is on the upper boundary, namely at $r = 25$. Here is the real problem. The restriction that r must be in the range 1–25 has placed the integral in (1) in an interval of practically zero density. This may most easily be seen in Figure 1, which shows the joint posterior distribution of θ and r based on an inverse gamma prior with parameters $a = 6.000$ and $b = .0317$ for θ and a uniform prior for r in the interval from 1 to 150. In this figure, the outer contour corresponds to the 99%

Table 1. The Correct Values for a and b Corresponding to the Choices for L, U, and P_0 of Piegorsch and Gladen (1986)

L	U	a	b
.5	3.0	2.392	.4481
.5	4.0	1.858	.5440
1.0	2.0	14.039	.0526
1.0	2.5	8.187	.0833
1.0	3.0	5.803	.1108
1.0	4.0	3.773	.1568
1.5	2.5	25.543	.0207
1.5	3.0	14.039	.0351
1.5	4.0	7.191	.0619

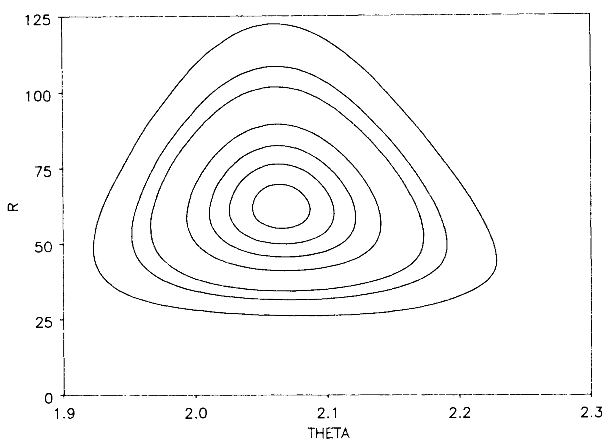

Figure 1. Contour Plot of the Joint Posterior Distribution of θ and r (.10, .30, .50, .70, .90, .95, .99 HPD intervals).

highest posterior density HPD region, and it is clear that values of r less than 25 are extremely unlikely on the basis of the authors' data. This is also to be seen in Figure 2, which shows the corresponding marginal posterior distribution of r. It may be that there are cases for which an upper limit of 25 is reasonable for r—the authors said, "The value $\Psi = 25$ was thought to be an acceptable upper bound on r" (p. 272)—but this is certainly not the case for these data.

A more reasonable choice for an upper bound on r has further consequences. First, there is no need to use quadruple precision arithmetic. The calculations that I have carried out were all in single precision, and no instabilities were experienced. Second, integration over a more realistic region leads to considerably shorter intervals for θ. Taking the prior above the calculated HPD limits for θ are 1.975 and 2.161, which should be compared with the authors' given limits of 1.42 and 3.53.

I have two final comments. First, the preceding considerations apply equally well to the double uniform prior and to the pseudo-Bayes approach considered by the authors. Second, as the authors rightly

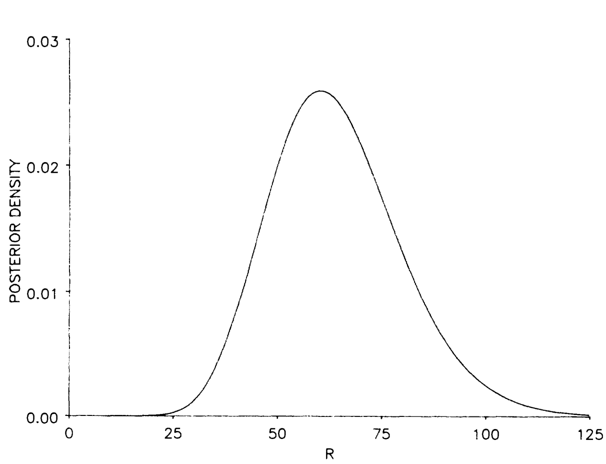

Figure 2. Posterior Distribution of r.

point out, the posterior distribution of θ given the data and the value of r is inverse gamma, and it is worthwhile to consider the proposal of Bartolucci and Dickey (1977) to approximate the marginal posterior distribution of θ, given that r takes its marginal modal value. This proposal is attractive, since programs to deal with inverse gamma distributions are readily available. By looking at Figure 2, it might be thought that the posterior distribution of r is not "highly concentrated about its mode." This is true; nonetheless the HPD limits for θ using this approach are 1.976 and 2.160, and the approximation is worthy of consideration.

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RESPONSE TO A. P. GRIEVE

Grieve is certainly correct in noting an algorithmic error in our presentation. His Step 5 should replace ours, although we would continue to advocate the use of a lower limit, s, for θ instead of the possibly arbitrary value 10^{-6}.

In discussing the computational difficulties associated with evaluation of the posterior density for θ from our example, Grieve makes two points. The first is that the prior belief we had about the parameter r, which we quantified as a uniform distribution on [1, 25], does not accord well with the values for r suggested by the data. The second is that different prior distributions on r, such as a uniform distribution on [1, 150], lead to posterior distributions that are much more readily computed. We do not dispute either statement. In fact, we found Grieve's observations useful, because they shed more light on the complexities of the computations, and we were happy to see that our exposition had attracted such interest.

We do, however, dispute the conclusion that the difficulties were largely of our own making. Statisticians employing Bayesian procedures start with prior beliefs, collect data, and then revise their beliefs in light of these data. In some cases, the data will simply confirm prior beliefs, but in other cases the data will suggest considerable changes. The example in our article falls into the latter category, as would many other cases in actual practice. A procedure that only works well when the prior beliefs are confirmed would not be of much use; if one knows the answer ahead of time, there is no point in conducting the experiment. Thus although our prior ideas about the value of r may well have been characterized as "unhappy"—and this is certainly an accurate description of our emotions while battling with the programming difficulties—they were neither unreasonable nor unrealistic. We stand by our conclusion that the procedure involving the evaluation of our Equation (2.2) is of limited usefulness.

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